Global well-posedness and backward uniqueness of stochastic 3D Burgers equation in $\mathbb{L}^2(\mathbb{T}^3;\mathbb{R}^3)$

Guoli Zhou

With Zdzislaw.Brzezniak, Zhao Dong and Shijia Zhang

Tianjin University

2023.7.05

Outline

Outline

1. Introduction and main results to stochastic 3D Burgers equation (BE)

2. Local well-posedness of random 3D BE (5)
   - Local well-posedness of difference equation (6)
   - Local well-posedness of random 3D BE (5)

3. Global well-posedness of random 3D BE (5)
   - Local well-posedness of a regularization of 3D BE (7)
   - Global well-posedness of the regularization of 3D BE (7)
   - Global well-posedness of random 3D BE (5)
   - Backward uniqueness of the solutions to random 3D BE (5)

3D Burgers equation

Let $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ be the 3-dimensional torus, 3D BE refers to the following equation:

$$\begin{cases} d\mathbf{u}(t) - \Delta\mathbf{u}(t)dt + (\mathbf{u}\cdot\nabla\mathbf{u})(t)dt = 0, \quad \text{on } [0,T]\times\mathbb{T}^3, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0, \quad \mathbf{x} = (x_1,x_2,x_3)\in\mathbb{T}^3. \end{cases}$$

Known results:

- Kiselev and Ladyzhenskaya (1957, [5]) : global well-posedness for 3D BE in $L^\infty([0,T];L^\infty(\mathcal{O}))\cap L^2([0,T];H_0^1(\mathcal{O}))$.
- Robinson, Rodrigo, and Sadowski (2016, [7]): the global well-posedness of weak solution in $\mathbb{H}^{1/2}(\mathbb{T}^3)$ of 3D deterministic BE, but the global well-posedness in $\mathbb{L}^2(\mathbb{T}^3)$ is unknown .

stochastic 3D Burgers equation

Let $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ be the 3-dimensional torus, consider stochastic 3D BE,

$$\begin{cases} d\mathbf{u}(t) - \Delta\mathbf{u}(t)dt + (\mathbf{u}\cdot\nabla\mathbf{u})(t)dt = \mathbf{u}(t)\circ b dB(t) + d\mathbf{w}(t), & \text{on } [0,T]\times\mathbb{T}^3, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0, \quad \mathbf{x} = (x_1,x_2,x_3)\in\mathbb{T}^3. \end{cases} \tag{1}$$

where: $B(t)$ is 1-dim BM; $b\in\mathbb{R}$; $\mathbf{w}(t) = \sum_{\mathbf{k}\in\mathbb{Z}^3}\lambda_{\mathbf{k}}\exp(i\mathbf{x}\cdot\mathbf{k})\tilde{\mathbf{B}}_{\mathbf{k}}(t)$, $\tilde{\mathbf{B}}_{\mathbf{k}}(t)$ is 3-dim BM.

Known results :

- Brzezniak, Goldys, Neklyudov (2014, [1]) : the global existence and uniqueness of mild solutions in $L^p(\mathbb{T}^3)$ and $L^p(\mathbb{R}^3), p>3,$ for the stochastic 3D BE with additive noise.
- Dong, Guo, Wu, Zhou (2023, [3, 4]) : the global well-posedness and ergodicity of weak solutions in $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ for the stochastic 3D BE with linear multiplicative noise.

Our aim : Global well-posedness of stochastic 3D BE (1) in $\mathbb{L}^2(\mathbb{T}^3)$.

Our answer : (1) is solved with $\mathbf{u}_0\in\mathbb{L}^2(\mathbb{T}^3)$ replaced with $\mathbf{u}_0^\omega\in\mathbb{L}^2(\mathbb{T}^3) a.s.$, $d\mathbf{w}$ replaced with $\mathbf{F}$.

The model and assumptions

That is, we consider

$$\begin{cases} d\mathbf{u}(t) - \Delta\mathbf{u}(t)dt + (\mathbf{u}\cdot\nabla\mathbf{u})(t)dt = \mathbf{F}(t)dt + \mathbf{u}(t)\circ bdB(t), & \text{on } [0,T]\times\mathbb{T}^3, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0^\omega \in \mathbb{L}^2(\mathbb{T}^3), & \mathbf{x} = (x_1,x_2,x_3)\in\mathbb{T}^3. \end{cases} \tag{2}$$

where $\mathbf{u}_0^\omega = \sum_{\mathbf{k}\in\mathbb{Z}^3} r_{\mathbf{k}}(\omega)\hat{\mathbf{u}}_{0,\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{x})$, $\mathbf{u}_0 = \sum_{\mathbf{k}\in\mathbb{Z}^3}\hat{\mathbf{u}}_{0,\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{x})$.

Note that $\mathbf{u}_0^\omega$ has the same regularity of $\mathbf{u}_0 \in \mathbb{L}^2(\mathbb{T}^3)$. That is, $\mathbf{u}_0^\omega \in \mathbb{L}^2(\mathbb{T}^3)$, *a.s.*, but not $\mathbf{u}_0^\omega \in \mathbb{H}^\sigma(\mathbb{T}^3)$, for some $\sigma > 0$ with positive probability.

Assumptions:

1 Let $\{r_{\mathbf{k}}(\omega)\}_{\mathbf{k}\in\mathbb{Z}^3}$ be a sequence of independent, 0 mean value, complex random variables on $(\Omega,\mathcal{F},\mathbb{P})$ such that for each $\mathbf{k}$, and some positive constant C

$$\sup_{\mathbf{k}\in\mathbb{Z}^3}\mathbb{E}(|r_{\mathbf{k}}(\omega)|^6) \le C \quad \text{and} \quad \bar{r}_{\mathbf{k}} = r_{-\mathbf{k}},\ \mathbf{k}\in\mathbb{Z}^3.$$

2 The external force $\mathbf{F}:[0,T]\times\mathbb{T}^3\to\mathbb{R}^3 \in \mathbb{L}^2([0,T];\mathbb{H}^1(\mathbb{T}^3))$.

Main results: Global well-posedness of (2)

Theorem. (Global well-posedness of stochastic 3D BE)

Let $\mathbf{u}_0 \in \mathbb{L}^2(\mathbb{T}^3)$ and $\mathbf{u}_0^\omega$ be independent of $(\mathbf{B}(t))_{t\in[0,T]}$. Under the assumptions, there $\exists!$ $\mathbf{u}$ to (2), i.e., $\mathbf{u} \in \mathbb{L}^\infty([0,T];\mathbb{L}^2(\mathbb{T}^3)) \cap \mathbb{L}^4([0,T];\mathbb{L}^6(\mathbb{T}^3)) \cap \mathbb{L}^2([0,T];\mathbb{H}^{1,3}(\mathbb{T}^3))$ and satisfies :

$$\langle \mathbf{u}(t), \eta\rangle = \langle \mathbf{u}_0^\omega, \eta\rangle - \int_0^t \langle(\Lambda \mathbf{u}(s), \Lambda\eta\rangle - \int_0^t \langle(\mathbf{u}\cdot\nabla)\mathbf{u})(s), \eta\rangle ds + \int_0^t \langle \mathbf{F}(s), \eta\rangle ds + \int_0^t \langle \eta, \mathbf{u}(s)\circ dB(s)\rangle,$$

on $[0,T]$, where $\eta \in D(\Lambda^2)$. Moreover, the backward uniqueness also holds for $\mathbf{u}$.

$$\mathbf{F} \Rightarrow d\mathbf{w}, \mathbf{w}(t) := \Big(\sum_{j\in\mathbb{N}} \lambda_j \exp(ix_1\cdot j)\tilde{B}_j^1(t), \sum_{k\in\mathbb{N}} \lambda_k \exp(ix_2\cdot k)\tilde{B}_k^2(t), \sum_{\mathbf{k}\in\mathbb{Z}^3} \lambda_{\mathbf{k}} \exp(i\mathbf{x}\cdot\mathbf{k})\tilde{B}_{\mathbf{k}}^3(t)\Big)$$

Under assumption $\sum_{\mathbf{k}\in\mathbb{Z}^3} |\lambda_{\mathbf{k}}|^2|\mathbf{k}|^6 < \infty$, we have

Theorem. (Global well-posedness of stochastic 3D BE with $\mathbf{F} \Rightarrow d\mathbf{w}$)

$\exists!$ $\mathbf{u}$ to (2), i.e., $\mathbf{u} \in \mathbb{L}^\infty([0,T];\mathbb{L}^2(\mathbb{T}^3)) \cap \mathbb{L}^4([0,T];\mathbb{L}^6(\mathbb{T}^3)) \cap \mathbb{L}^2([0,T];\mathbb{H}^{1,3}(\mathbb{T}^3))$ and satisfies

$$\langle \mathbf{u}(t), \eta\rangle = \langle \mathbf{u}_0^\omega, \eta\rangle - \int_0^t \langle(\Lambda \mathbf{u}(s), \Lambda\eta\rangle - \int_0^t \langle(\mathbf{u}\cdot\nabla)\mathbf{u})(s), \eta\rangle ds + \int_0^t \langle \eta, \mathbf{u}(s)\circ dB(s)\rangle + \langle \mathbf{w}(t), \eta\rangle,$$

on $[0,T]$, where $\eta \in D(\Lambda^2)$. Moreover, the backward uniqueness also holds for $\mathbf{u}$.

A brief description of the route to solving (2)

Let $\alpha(t) = \exp(bB(t)), t \geq 0$, and set

$$\mathbf{z}_0 = \mathbf{z}_0(t) = \mathbf{z}_0(t, \mathbf{x}) = \int_0^t e^{(t-s)\Delta} \alpha^{-1}(s) \mathbf{F}(s) ds, (t, \mathbf{x}) \in [0, T] \times \mathbb{T}^3, \ \ T > 0.$$

Let $\mathbf{u} = \alpha \tilde{\mathbf{v}} + \alpha \mathbf{z}_0$. Then the stochastic 3D Burgers equation (2) with random initial value $\mathbf{u}_0^\omega$ can be transformed into the following random case:

$$\begin{aligned} &\partial_t \tilde{\mathbf{v}}(t) - \Delta \tilde{\mathbf{v}}(t) + \alpha(t)(\tilde{\mathbf{v}} + \mathbf{z}_0) \cdot \nabla(\tilde{\mathbf{v}} + \mathbf{z}_0)(t) = 0, \ \text{on} \ (0, T] \times \mathbb{T}^3, \\ &\tilde{\mathbf{v}}(0) = \mathbf{u}_0^\omega \in \mathbb{L}^2(\mathbb{T}^3). \end{aligned} \tag{3}$$

For solving (3), we decompose it into a nonlinear partial differential equation with zero initial data (4) $\mathbf{v}(t) := \tilde{\mathbf{v}}(t) - e^{\Delta t} \mathbf{u}_0^\omega$ and a linear part $e^{\Delta t} \mathbf{u}_0^\omega$.

$$
\begin{aligned}
&\partial_t \mathbf{v}(t) - \Delta \mathbf{v}(t) + \alpha(t)(\mathbf{v}+\mathbf{z})\cdot\nabla(\mathbf{v}+\mathbf{z})(t,\mathbf{x}) = 0, \text{ on } [0,T]\times\mathbb{T}^3, \\
&\mathbf{v}(0) = 0,
\end{aligned} \tag{4}
$$

where $\mathbf{z} := \mathbf{z}(t) := \mathbf{z}(t,\mathbf{u}_0^\omega) = \mathbf{z}_0(t) + e^{\Delta t}\mathbf{u}_0^\omega$ satisfies

$$
\begin{aligned}
&d\mathbf{z}(t) - \Delta \mathbf{z}(t)dt = \alpha^{-1} d\mathbf{w}(t), \text{ on } [0,T]\times\mathbb{T}^3, \\
&\mathbf{z}(0) = \mathbf{u}_0^\omega \in \mathbb{L}^2(\mathbb{T}^3).
\end{aligned}
$$

From Proposition 1, the randomization $\mathbf{u}_0^\omega$ of the initial data $\mathbf{u}_0$ can improve the integrability of $\mathbf{z}$. Consequently, it contributes to the existence of the local solution $(\mathbf{v}, T_\omega)$ of (4). Observing the solution $\tilde{\mathbf{v}}$ of (3) satisfies:

$$
\tilde{\mathbf{v}}(t) = \mathbf{v}(t) + e^{\Delta t}\mathbf{u}_0^\omega, t \in [0, T_\omega],
$$

the local well-posedness of (3) is obtained. In view of the parabolic structure of (3), we further know

$$
\tilde{\mathbf{v}}(T_\omega) \in \mathbb{H}^2(\mathbb{T}^3), \text{a.s.}.
$$

A brief description of the route to solving (2)-Continued

The next question is extending the local solution $(\tilde{\mathbf{v}}, T_\omega)$ to being a global one. A natural approach is to use the maximum principle. However, the maximum principle can not be directly applied to random 3D Burgers equation (3). Hence, we introduce a regularization system (5) which has global well-posedness.

$$\partial_t \mathbf{v}_m^{\epsilon_n}(t,\mathbf{x}) - \Delta \mathbf{v}_m^{\epsilon_n}(t,\mathbf{x}) + \alpha^{\epsilon_n}(t)(\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_{0,m}^{\epsilon_n}) \cdot \nabla(\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_{0,m}^{\epsilon_n})(t,\mathbf{x}) = 0, [0, T] \times \mathbb{T}^3, \tag{5}$$
$$\mathbf{v}_m^{\epsilon_n}(0,\mathbf{x}) = \tilde{\mathbf{v}}(T_\omega) \in \mathbb{H}^2(\mathbb{T}^3), \quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{T}^3,$$

where α^{ϵ_n} and $\mathbf{z}_{0,m}^{\epsilon_n}$ are smooth with respect to $(t,\mathbf{x}) \in [0, T] \times \mathbb{T}^3$ so that the maximum principle is available for (5). By establishing

$$\mathbf{v}_m^{\epsilon_n} \rightrightarrows \tilde{\mathbf{v}} \text{ uniformly on } [T_\omega, \xi) \times \mathbb{T}^3, \text{ as } n,\ m \to \infty,$$

where ξ is the maximum existence time of $\tilde{\mathbf{v}}$, we achieve the global well-posedness of (3), which leads to the global well-posedness of (2).

A brief description of the route to solving (2)-Continued

$$\underset{\text{difference equation}}{\overset{\text{local existence}}{\left(4\right)}} \Longrightarrow \underset{\text{random 3D BE}}{\overset{\text{local well-posedness}}{\left(3\right)}} \Longrightarrow \underset{\text{regularization-(3)}}{\overset{\text{local well-posedness}}{\left(5\right)}} \Longrightarrow \underset{\text{regularization-(3)}}{\overset{\text{global well-posedness}}{\left(5\right)}}$$

$$\Longrightarrow \underset{\text{random 3D BE}}{\overset{\text{global well-posedness}}{\left(3\right)}} \Longrightarrow \underset{\text{stochastic 3D BE}}{\overset{\text{global well-posedness}}{\left(2\right)}}.$$

Notations and setup

- For $1 \leq p \leq \infty$, $\mathbb{L}^p(\mathbb{T}^3)$ is the Lebesgue space $\mathbb{L}^p(\mathbb{T}^3;\mathbb{R}^3)$ with the norm $|\cdot|_p$. When$p = 2$, $\langle \cdot, \cdot \rangle$ represents the inner product in $\mathbb{L}^2(\mathbb{T}^3)$. For $s \geq 0$, we introduce an operator Λ^s acting on $\mathbb{H}^s(\mathbb{T}^3)$ which is a Sobolev space $\mathbb{H}^s(\mathbb{T}^3;\mathbb{R}^3)$.
- Assuming $f \in \mathbb{H}^s(\mathbb{T}^3)$ with the Fourier series and norm

$$f(x) = \sum_{k\in\mathbb{Z}^3} \hat{f}_k e^{ik\cdot x} \in \mathbb{H}^s(\mathbb{T}^3), \ \|f\|_{\mathbb{H}^s} = \Big(\sum_{k\in\mathbb{Z}^3}(1+|k|^{2s})|\hat{f}_k|^2\Big)^{1/2} < \infty.$$

- Define

$$\Lambda^s f(x) = \sum_{k\in\mathbb{Z}^3} |k|^s \hat{f}_k e^{ik\cdot x} \in \mathbb{L}^2(\mathbb{T}^3).$$

Obviously, $\Lambda^2 = -\Delta$. Denote by $\|\cdot\|_s$ the seminorm $|\Lambda^s \cdot|_2$, then the Sobolev norm $\|\cdot\|_{\mathbb{H}^s}$ of $\mathbb{H}^s(\mathbb{T}^3)$ is equivalent to $|\cdot|_2 + \|\cdot\|_s$.

Definitions

Definition (Local weak/mild solutions to equation (4))

1. $(\mathbf{v}, T_\omega)$ is a local weak pathwise solution to (4) if there exists a positive random variable T_ω such that for almost all $\omega \in \Omega$, $\mathbf{v} \in \mathbb{L}^\infty([0, T_\omega]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T_\omega]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ with $\frac{d\mathbf{v}}{dt} \in \mathbb{L}^1([0, T_\omega]; \mathbb{H}^{-\frac{1}{2}}(\mathbb{T}^3))$ and for almost every $t \in [0, T_\omega]$ and for all $\eta \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$,

$$\left\langle \partial_t \mathbf{v}, \eta \right\rangle + \left\langle \Lambda^{\frac{3}{2}} \mathbf{v}, \Lambda^{\frac{1}{2}} \eta \right\rangle + \alpha \left\langle (\mathbf{v} + \mathbf{z}) \cdot \nabla (\mathbf{v} + \mathbf{z}), \eta \right\rangle = 0,$$

and

$$\lim_{t \to 0+} \mathbf{v}(t) = 0, \text{ weakly in the } \mathbb{L}^2(\mathbb{T}^3), \text{ a.s..}$$

2. $(\mathbf{v}, T_\omega)$ is a local mild pathwise solution to (4) if there exists a positive random variable T_ω such that for almost all $\omega \in \Omega$, $\mathbf{v} \in \mathbb{L}^\infty([0, T_\omega]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T_\omega]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ and for $t \in [0, T_\omega]$

$$\mathbf{v}(t) = \int_0^t \alpha^{-1}(s) e^{(t-s)\Delta} \Big((\mathbf{v} + \mathbf{z}) \cdot \nabla (\mathbf{v} + \mathbf{z}) \Big)(s) ds.$$

Outline

Regularity of stochastic heat flow

Proposition 1

For $T \in (0,1]$ and $p \geq 1$

$$\|\mathbf{z}(t,\mathbf{u}_0^\omega)\|_{\mathbb{L}^6(\Omega;\mathbb{L}^4([0,T];\mathbb{L}^6(\mathbb{T}^3)))} \leq c\|\mathbf{u}_0\|_{\mathbb{L}^2(\mathbb{T}^3)} T^{\frac{1}{4}} + cT^{\frac{1}{4}},$$
$$\|\nabla\mathbf{z}(t,\mathbf{u}_0^\omega)\|_{\mathbb{L}^6(\Omega;\mathbb{L}^2([0,T];\mathbb{L}^3(\mathbb{T}^3)))} \leq c\left\|\hat{\mathbf{u}}_{0,\mathbf{k}}\sqrt{1-\exp(-2|\mathbf{k}|^2T)}\right\|_{l^2} + cT^{\frac{1}{2}},$$

where c is independent of T. Consequently,

$$\mathbb{P}(\mathbb{S}_{\lambda,T,\mathbf{u}_0}) \leq c\lambda^{-6}\left(\|\mathbf{u}_0\|_{\mathbb{L}^2(\mathbb{T}^3)} T^{\frac{1}{4}} + T^{\frac{1}{4}} + T^{\frac{1}{2}} + \left\|\hat{\mathbf{u}}_{0,\mathbf{k}}\sqrt{1-\exp(-2|\mathbf{k}|^2T)}\right\|_{l^2}\right)^6,$$

where both λ and c are independent of T, and

$$\mathbb{S}_{\lambda,T,\mathbf{u}_0} = \{\omega \in \Omega : \|\mathbf{z}(t,\mathbf{u}_0^\omega)\|_{\mathbb{L}^4([0,T];\mathbb{L}^6(\mathbb{T}^3))} + \|\nabla\mathbf{z}(t,\mathbf{u}_0^\omega)\|_{\mathbb{L}^2([0,T];\mathbb{L}^3(\mathbb{T}^3))} \geq \lambda\}.$$

Local mild solutions to Difference equation (4)

Theorem.

Let $\mathbf{u}_0^\omega \in \mathbb{L}^2(\mathbb{T}^3)$, a.s.. Then for almost all $\omega \in \Omega$ there exists a unique local mild solution $(\mathbf{v}, T_\omega)$ to (4). More precisely, there exists $C > 0$, for arbitrary $T \in (0, 1]$ and event $\Omega_T \in \mathcal{F}$ such that

$$\mathbb{P}(\Omega_T) \geq 1 - C\left(\|\mathbf{u}_0\|_{\mathbb{L}^2(\mathbb{T}^3)} T^{\frac{1}{4}} + T^{\frac{1}{4}} + T^{\frac{1}{2}} + \left\|\hat{\mathbf{u}}_{0,\mathbf{k}}\sqrt{1 - \exp(-2|\mathbf{k}|^2 T)}\right\|_{l^2}\right)^6,$$

and for every $\omega \in \Omega_T$, there exits a unique local mild solution $(\mathbf{v}, T)$ to (4) belongs to $C([0, T]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$.

Sketch proof

Proof. We will use contraction principle to prove the theorem. So, we define a map

$$L : \mathbf{v}(t,\mathbf{x}) \mapsto -\int_0^t e^{(t-s)\Delta}(\alpha^{-1}(s)(\mathbf{v}(s,\mathbf{x})+\mathbf{z}(s,\mathbf{u}_0^\omega))\cdot\nabla(\mathbf{v}(s,\mathbf{x})+\mathbf{z}(s,\mathbf{u}_0^\omega)))ds.$$

where $\mathbf{v} \in X := \mathbb{L}^\infty([0,T];\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathbb{L}^2([0,T];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ with the norm

$$\|\mathbf{f}\|_X = \|\mathbf{f}\|_{\mathbb{L}^\infty([0,T];\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))} + \|\nabla\mathbf{f}\|_{\mathbb{L}^2([0,T];\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))}.$$

For $\omega \in \mathbb{S}^c_{\lambda,T,\mathbf{u}_0}$ and $T \in (0,1]$, we have

$$\|\mathbf{z}(t,\mathbf{u}_0^\omega)\|_{\mathbb{L}^4([0,T];\mathbb{L}^6(\mathbb{T}^3))} + \|\nabla\mathbf{z}(t,\mathbf{u}_0^\omega)\|_{\mathbb{L}^2([0,T];\mathbb{L}^3(\mathbb{T}^3))} < \lambda.$$

For constants p and q satisfying $\frac{2}{p}+\frac{3}{q}=3$ with $q \in (\frac{3}{2},2]$, we have

$$\left\|\int_0^t e^{(t-s)\Delta}\mathbf{f}(s,\mathbf{x})ds\right\|_{\mathbb{L}^\infty([0,T];\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))\cap\mathbb{L}^2([0,T];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))} \le c\|\mathbf{f}\|_{\mathbb{L}^p([0,T];\mathbb{L}^q(\mathbb{T}^3))},$$

where the constant c is independent of T.

Sketch proof—Continued

For $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{L}^\infty([0, T]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$, we have

$$\|L(\mathbf{v}_i)\|_X \leq c(\lambda^2 + \|\mathbf{v}\|_X^2),$$

and

$$\|L(\mathbf{v}_1) - L(\mathbf{v}_2)\|_X \leq c\|\mathbf{v}_1 - \mathbf{v}_2\|_X(\lambda + \|\mathbf{v}_1\|_X + \|\mathbf{v}_2\|_X),$$

where c is independent of λ and T. Then on $X_\lambda := \{\mathbf{f} \in X : \|\mathbf{f}\|_X \leq 2c\lambda^2\}$, there exists λ satisfying

$$c\lambda^2 + c(2c\lambda^2)^2 \leq 2c\lambda^2 \text{ and } c\lambda + 2c(2c\lambda^2) \leq \frac{3}{4}.$$

which implies L satisfies the contraction principle on X_λ. Hence, the local existence of mild solution $\mathbf{v}$ follows. Note that

$$\mathbb{P}(\mathbb{S}^c_{\lambda, \frac{1}{n}, \mathbf{u}_0}) \uparrow 1, \ as\ n \uparrow \infty,$$

the result is proven.

Local weak solutions to difference equation (4) and random 3D BE (3)

Lemma. (Local weak solutions to difference equation (4))

The existence of local mild solutions and local weak solutions to (4) are equivalent.

Note that $\tilde{\mathbf{v}} = \mathbf{v} + e^{\Delta t}\mathbf{u}_0^{\omega}$, where $\mathbf{v}$ is a local weak solution to (4). Since

$$\mathbf{v} \in \mathbb{L}^{\infty}([0, T_{\omega}]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T_{\omega}]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$$

$$\text{and } e^{\Delta t}\mathbf{u}_0^{\omega} \in \mathbb{L}^{\infty}([0, T_{\omega}]; \mathbb{L}^2(\mathbb{T}^3)) \cap \mathbb{L}^4([0, T_{\omega}]; \mathbb{L}^6(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T_{\omega}]; \mathbb{H}^{1,3}(\mathbb{T}^3)),$$

we get

$$\tilde{\mathbf{v}} \in \mathbb{L}^{\infty}([0, T_{\omega}]; \mathbb{L}^2(\mathbb{T}^3)) \cap \mathbb{L}^4([0, T_{\omega}]; \mathbb{L}^6(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T_{\omega}]; \mathbb{H}^{1,3}(\mathbb{T}^3))$$

and

Theorem. (Local weak solutions to random 3D Burgers equation (3))

For $\mathbf{u}_0 \in \mathbb{L}^2(\mathbb{T}^3)$, then there exists a unique local weak solution $(\tilde{\mathbf{v}}, T_{\omega})$ to (3) a.s..

Outline

Steps to establish the global well-posedness of random 3D BE in $\mathbb{L}^2(\mathbb{T}^3)$

Step 1: Establishing a regularization system (7) of random 3D Burgers equation and obtaining its local existence of classical solutions .

Step 2: Applying the maximum principle to the regularization system (7) to obtain the *a priori* estimates, which implies the global existence of the classical solutions.

Step 3: Proving the solutions to (7) converge to (3) uniformly to establish the global existence and uniqueness of the weak solutions to (3) in $\mathbb{L}^2(\mathbb{T}^3)$.

Step 1: Local well-posedness of regularization system (7)

Project $\mathbf{F}(t)$ to $\mathbf{F}_m(t)$, $m \in \mathbb{N}$

$$\mathbf{F}_m(t) = \sum_{\mathbf{k}\in\mathbb{Z}^3, |\mathbf{k}|\le m} \hat{\mathbf{F}}_{\mathbf{k}}(t) \exp(i\mathbf{x}\cdot\mathbf{k}),\ \hat{\mathbf{F}}_{\mathbf{k}}(t) = \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} \exp(i\mathbf{x}\cdot\mathbf{k})\mathbf{F}(t,\mathbf{x})d\mathbf{x}. \quad (t,\mathbf{x}) \in [0,T]\times\mathbb{T}^3.$$

Denote by B^{ϵ_n} the mollification of B, then $\alpha^{\epsilon_n}(t) := \exp\left(B^{\epsilon}(t)\right) \in C^\infty([0,T];\mathbb{R})$. Similarly, define $\mathbf{F}_m^{\epsilon_n}$. Let

$$\mathbf{z}_{0,m}^{\epsilon_n}(t,\mathbf{x}) = \int_0^t e^{(t-s)\Delta}(\alpha^{\epsilon_n})^{-1}(s)\mathbf{F}_m^{\epsilon_n}(s)ds.$$

Obviously, $\mathbf{z}_{0,m}^{\epsilon_n} \in C^\infty([0,T]\times\mathbb{T}^3;\mathbb{R}^3)$. By technique of harmonic analysis we have $\left\| \int_0^t e^{(t-s)\Delta}\mathbf{f}(s,\mathbf{x})ds \right\|_{\mathbb{L}^\infty([0,T];\mathbb{H}^2(\mathbb{T}^3))} \le c\|\mathbf{f}\|_{\mathbb{L}^2([0,T];\mathbb{H}^1(\mathbb{T}^3))}$, which implies $n, m \to \infty$, we have

$$|\mathbf{z}_{0,m}^{\epsilon_n} - \mathbf{z}_0|_{\mathbb{L}^\infty([0,T];\mathbb{H}^2(\mathbb{T}^3))} \to 0,\ \ |\alpha^{\epsilon_n}(t) - \alpha(t)|_{\mathbb{L}^\infty([0,T])} \to 0. \tag{6}$$

To prove the global existence of $\tilde{\mathbf{v}}$, we need to introduce a regularization system (7):

$$\partial_t\mathbf{v}_m^{\epsilon_n}(t,\mathbf{x}) - \Delta\mathbf{v}_m^{\epsilon_n}(t,\mathbf{x}) + \alpha^{\epsilon_n}(t)(\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_{0,m}^{\epsilon_n})\cdot\nabla(\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_{0,m}^{\epsilon_n})(t,\mathbf{x}) = 0,\ \text{on } [0,T]\times\mathbb{T}^3, \tag{7}$$
$$\mathbf{v}^{\epsilon_n}(0,\mathbf{x}) = \mathbf{v}_0, \quad \mathbf{x} = (x_1,x_2,x_3) \in \mathbb{T}^3.$$

Step 1: Local well-posedness of regularization system (7)

Proposition. (Local strong solutions of regularization system (7))

For $\mathbf{v}_0 \in \mathbb{H}^2(\mathbb{T}^3)$, there exists a unique strong solution $(\mathbf{v}_m^{\epsilon_n}, T_\omega^{0,\epsilon})$ to (7) such that $\mathbf{v}_m^{\epsilon_n} \in C([0, T_\omega^{0,\epsilon}]; \mathbb{H}^2(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T_\omega^{0,\epsilon}]; \mathbb{H}^3(\mathbb{T}^3))$.

Let $(\mathbf{v}_m^{\epsilon_n}, \xi^{\epsilon_n})$ be the maximum strong solution to (7). By establishing the uniform *a priori* estimates for the Faedo-Galerkin approximation $(\mathbf{v}_m^{\epsilon_n})_N$: for any $0 < T_\omega^{1,\epsilon} < T_\omega^{2,\epsilon} < \xi^\epsilon$,

$$\sup_N \|(\mathbf{v}_m^{\epsilon_n})_N\|_{\mathbb{H}^1([T_\omega^{1,\epsilon}, T_\omega^{2,\epsilon}];\mathbb{H}^4)} < \infty, \text{ and } \sup_N \|(\mathbf{v}_n^{\epsilon})_N\|_{\mathbb{H}^2([T_\omega^{1,\epsilon}, T_\omega^{2,\epsilon}];\mathbb{H}^2)} < \infty, \ a.s.,$$

Proposition. (Local classical solutions to regularization system (7))

For $\mathbf{v}_0 \in \mathbb{H}^2(\mathbb{T}^3)$, the maximum strong solution $(\mathbf{v}_m^{\epsilon_n}, \xi^{\epsilon_n})$ to (7) is classical on $(0, \xi^{\epsilon_n})$, i.e., $\mathbf{v}_m^{\epsilon_n} \in C^{0,2}((0, \xi^{\epsilon_n}) \times \mathbb{T}^3; \mathbb{R}^3) \cap C^{1,0}((0, \xi^{\epsilon_n}) \times \mathbb{T}^3; \mathbb{R}^3)$.

Step 2: Global well-posedness of regularization system (7)

Proposition. (Global existence of solutions to system (7))

The maximum solution $(\mathbf{v}_m^{\epsilon_n}, \xi^{\epsilon_n})$ with $\mathbf{v}_0 \in \mathbb{H}^2(\mathbb{T}^3)$ satisfies the a priori estimates

$$\sup_{n,m\in\mathbb{N}} \sup_{t\in[0,\xi^{\epsilon_n}\wedge T)} |\mathbf{v}_m^{\epsilon_n}(t)|_{\mathbb{L}^\infty(\mathbb{T}^3)} \le C(\|\mathbf{v}_0\|_{\mathbb{H}^2(\mathbb{T}^3)}, |\alpha^{-1}|_{\mathbb{L}^\infty([0,T])}, |\mathbf{F}|_{\mathbb{L}^2([0,T];\mathbb{H}^1(\mathbb{T}^3))}) < \infty,$$

which implies the strong solution $\mathbf{v}_m^{\epsilon_n}$ is global.

Proof.
The global existence proof of $\mathbf{v}_m^{\epsilon_n}$ based on the a priori estimates.

$$\frac{1}{2}\partial_t\|\mathbf{v}_m^{\epsilon_n}\|_1^2 + \|\mathbf{v}_m^{\epsilon_n}\|_2^2 \le |\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_{0,m}^{\epsilon_n}|_{\mathbb{L}^\infty(\mathbb{T}^3)}\|\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_{0,m}^{\epsilon_n}\|_1\|\mathbf{v}_m^{\epsilon_n}\|_2.$$

By the Gronwall inequality,

$$\sup_{n,m\in\mathbb{N}} \sup_{t\in[0,\xi^{\epsilon_n}\wedge T)} \left(\|\mathbf{v}_m^{\epsilon_n}(t)\|_1^2 + \int_0^t \|\mathbf{v}_m^{\epsilon_n}(s)\|_2^2 ds\right) \le C(\|\mathbf{v}_0\|_{\mathbb{H}^2(\mathbb{T}^3)}, |\alpha^{-1}|_{\mathbb{L}^\infty([0,T])}, |\mathbf{F}|_{\mathbb{L}^2([0,T];\mathbb{H}^1(\mathbb{T}^3))}) := C(\mathbf{v}_0, \mathbf{F}, \alpha)$$

Similarly, one can prove

$$\sup_{n,m\in\mathbb{N}} \sup_{t\in[0,\xi^{\epsilon_n}\wedge T)} \|\mathbf{v}_m^{\epsilon_n}(t)\|_2^2 \le C(\mathbf{v}_0, \mathbf{F}, \alpha) < \infty. \Rightarrow \xi^{\epsilon_n} = \infty, \text{a.s.}.$$

Step 3: Global well-posedness of random 3D BE (5)

Proposition. (Global well-posedness of random 3D BE (3))

Let $\mathbf{u}_0^\omega \in \mathbb{L}^2(\mathbb{T}^3)$. Then, the maximum weak solution $(\tilde{\mathbf{v}}, \xi)$ to (3) is global.

Proof. Consider random 3D BE (3) and regularization system (7) on $[T_\omega, \xi)$ with $\mathbf{v}_m^{\epsilon_n}(T_\omega) = \tilde{\mathbf{v}}(T_\omega) \in \mathbb{H}^2(\mathbb{T}^3)$. Define $\mathbf{w}_m^{\epsilon_n} = \mathbf{v}_m^{\epsilon_n} - \tilde{\mathbf{v}}$. Then, on $[T_\omega, \xi)$, we have

$$\begin{aligned}\partial_t \mathbf{w}_m^{\epsilon_n} &- \Delta \mathbf{w}_m^{\epsilon_n} + [\alpha^{\epsilon_n} - \alpha](\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_0^{\epsilon_n}) \cdot \nabla(\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_{0,m}^{\epsilon_n}) \\ &+ \alpha[\mathbf{w}_m^{\epsilon_n} + (\mathbf{z}_{0,m}^{\epsilon_n} - \mathbf{z}_0)] \cdot \nabla(\mathbf{v}_m^{\epsilon_n} + \mathbf{z}_{0,m}^{\epsilon_n}) + \alpha(\tilde{\mathbf{v}} + \mathbf{z}_0) \cdot \nabla[\mathbf{w}_m^{\epsilon_n} + (\mathbf{z}_{0,m}^{\epsilon_n} - \mathbf{z}_0)] = 0.\end{aligned}$$

By showing that

$$\sup_{t\in[T_\omega,\xi)} \|\mathbf{w}_m^{\epsilon_n}(t)\|_2^2 I_{\{\xi<\infty\}} \to 0, \quad \text{as n, m} \to \infty, \text{ a.s..}$$

we get

$$\sup_{t\in[T_\omega,\xi)} |\tilde{\mathbf{v}}(t)|_{\mathbb{L}^\infty(\mathbb{T}^3)} I_{\{\xi<\infty\}} < C(\omega) < \infty, \text{ a.s..} \Longrightarrow \quad \xi = \infty, \text{ a.s..}$$

□

Backward uniqueness of the solutions to random 3D BE

Theorem. (Backward uniqueness of random 3D BE)

Suppose $\tilde{\mathbf{v}}^1$ and $\tilde{\mathbf{v}}^2$ are two weak solutions to (3) with random initial conditions $\mathbf{u}_1^\omega \in \mathbb{L}^2(\mathbb{T}^3)$ and $\mathbf{u}_2^\omega \in \mathbb{L}^2(\mathbb{T}^3)$ respectively. Then for $T > 0$,

$$\textit{if } \tilde{\mathbf{v}}^1(T) = \tilde{\mathbf{v}}^2(T), \text{ we have } \tilde{\mathbf{v}}^1(t) = \tilde{\mathbf{v}}^2(t) \text{ for } t \in [0, T], \text{ a.s..} \tag{8}$$

Proof. Let $\tilde{\mathbf{v}}^1$ and $\tilde{\mathbf{v}}^2$ be two solutions to (3). Denote by $\mathbf{v} := \tilde{\mathbf{v}}^1 - \tilde{\mathbf{v}}^2$, then we get

$$\log |\mathbf{v}(T)|_2^2 \geq -2K(T - t_0) - 2C(K+1) \int_{t_0}^{T} ||M(\tau)||^2_{L(\mathbb{H}^1, \mathbb{L}^2)} \, d\tau + \log |\mathbf{v}(t_0)|_2^2 ,$$

where $||M(s)||^2_{L(\mathbb{H}^1,\mathbb{L}^2)} = ||\tilde{\mathbf{v}}^1(s)||^2_{\mathbb{H}^2} + ||\tilde{\mathbf{v}}^2(s)||^2_{\mathbb{H}^2} + \|\mathbf{z}_0(s)\|^2_{\mathbb{H}^2}$. If $|\mathbf{v}(T)|_2^2 = 0$, then $|\mathbf{v}(t_0)|_2^2 = 0$ for arbitrary $t_0 \in (0, T)$. And $\mathbf{v}(0) = 0$ is derived from the weak continuity of the weak solutions to (3). □

References I

[1] Z. Brzezniak, B. Goldys and M. Neklyudov: *Multidimensional stochastic Burgers equation,* SIAM J. Math. Anal., 46 (2014), 871–889.

[2] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations I: Local existence theory*, Invent. Math., 173(2008), 449–475.

[3] Z.Dong, B.L. Guo, J.L.,Wu and G.L. Zhou: *Global well-posedness and rgularities of 3D stochastic Burgers equation,* SIAM Journal on Mathematical Analysis, 2023.

[4] Z.Dong, J.L.,Wu and G.L. Zhou: *Ergodicity of 3D stochastic Burgers equation,* Acta Mathematica Sinica, 2023.

[5] A. Kiselev and O. A. Ladyzhenskaya: *On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid* (Russian), Izv. Akad. Nauk SSSR. Ser. Mat., 21 (1957), 655–680.

[6] S.-E. A. Mohammed and Tusheng Zhang: *Stochastic Burgers equation with random initial velocities: a Malliavin calculus approach,* SIAM J.Math.Anal.,45(2013), 2396–2420.

[7] J.C. Robinson, J.L.Rodrigo and W.Sadowski: *Well-posedness for the diffusive 3D Burgers equations with initial data in $H^{1/2}$. In: Recent Progress in the Theory of the Euler and Navier-Stokes Equations,* 430(2016), Cambridge: Cambridge University Press, 137–153.

Many thanks for attention!